ON A SMOOTH COMPACTIFICATION OF $\mathrm{PSL}(n,\mathbb{C})/T$

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Abstract. Let T be a maximal torus of $\mathrm{PSL}(n,\mathbb{C})$. For $n \geq 4$, we construct a smooth compactification of $\mathrm{PSL}(n,\mathbb{C})/T$ as a geometric invariant theoretic quotient of the wonderful compactification $\overline{\mathrm{PSL}(n,\mathbb{C})}$ for a suitable choice of T–linearized ample line bundle on $\overline{\mathrm{PSL}(n,\mathbb{C})}$. We also prove that the connected component, containing the identity element, of the automorphism group of this compactification of $\mathrm{PSL}(n,\mathbb{C})/T$ is $\mathrm{PSL}(n,\mathbb{C})$ itself.

1. Introduction

Let G be a semisimple group of adjoint type over the field $\mathbb{C}$ of complex numbers. De Concini and Procesi in [DP] constructed a smooth projective variety $\overline{G}$ with an action of $G \times G$ such that

- the variety G equipped with the action of $G \times G$ given by the left and right translations is an open dense orbit of it, and
- the boundary $\overline{G} \setminus G$ is a union of $G \times G$ stable normal crossing divisors.

This variety $\overline{G}$ is known as the wonderful compactification of G.

Fix a maximal torus T of G. Consider the right action of T on $\overline{G}$, meaning the action of the subgroup $1 \times T \subset G \times G$. For a T–linearized ample line bundle $\mathcal{L}$ on $\overline{G}$, let $\overline{G}^{ss}_T(\mathcal{L})$ and $\overline{G}^{s}_T(\mathcal{L})$ denote respectively the loci of semistable and stable points of $\overline{G}$ (see [MFK, p. 30, p. 40]).

Our first main result (Proposition 3.3) says that there is a T–linearized ample line bundle $\mathcal{L}$ on $\overline{G}$ such that $\overline{G}^{ss}_T(\mathcal{L}) = \overline{G}^{s}_T(\mathcal{L})$.

For $G = \mathrm{PSL}(n,\mathbb{C})$, we show that there is a T–linearized ample line bundle $\mathcal{L}$ on $\overline{\mathrm{PSL}(n,\mathbb{C})}$ such that

- the GIT quotient $\overline{\mathrm{PSL}(n,\mathbb{C})}^{ss}_T(\mathcal{L})/\!\!/T$ is smooth, and
- the boundary $(\overline{\mathrm{PSL}(n,\mathbb{C})}^{ss}_T(\mathcal{L})/\!\!/T) \setminus (\mathrm{PSL}(n,\mathbb{C})/T)$ is a union of $\mathrm{PSL}(n,\mathbb{C})$ stable normal crossing divisors.

We further show that for $n \geq 4$, the connected component of the automorphism group of $\overline{\mathrm{PSL}(n,\mathbb{C})}^{ss}_T/\!\!/T$ containing the identity automorphism is $\mathrm{PSL}(n,\mathbb{C})$ (Theorem 4.1).

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2. Preliminaries and notation

In this section we recall some preliminaries and notation about Lie algebras and algebraic groups; see for example [Hu1] and [Hu2] for the details. Let G be a simple group of adjoint type of rank n over the field of complex numbers. Let T be a maximal torus of G and $B \supset T$ a Borel subgroup of G. Let $N_G(T)$ denote the normalizer of T in G. So $W := N_G(T)/T$ is the Weyl group of G with respect to T.

The Lie algebra of G will be denoted by $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of T. The set of roots of G with respect to T will be denoted by R. Let $R^+ \subset R$ be the set of positive roots with respect to B. Let

$$S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset R^+$$

be the set of simple roots with respect to B. The group of characters of T will be denoted by $X(T)$, while the group of one-parameter subgroups of T will be denoted by $Y(T)$. Let

$$\{\lambda_i \mid 1 \leq i \leq n\}$$

be the ordered set of one-parameter subgroups of T satisfying the condition that $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$, where

$$\langle -, - \rangle : X(T) \times Y(T) \longrightarrow \mathbb{Z}$$

is the natural pairing, and δ_{ij} is the Kronecker delta function. Let $\leq$ (respectively, $\geq$) be the partial order on $X(T)$ defined as follows: $\chi_1 \leq \chi_2$, (respectively, $\chi_1 \geq \chi_2$) if $\chi_2 - \chi_1$ (respectively, $\chi_1 - \chi_2$) is a linear combination of simple roots with non-negative integers as coefficients.

Let $(-, -)$ denote the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$. Let

$$\{\omega_j \mid 1 \leq j \leq n\}$$

be the ordered set of fundamental weights corresponding to S, in other words,

$$\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

For $1 \leq i \leq n$, let s_{α_i} denote the simple reflection corresponding to α_i.

The longest element of W corresponding to B will be denoted by w_0. Let

$$B^- = w_0 B w_0^{-1}$$

be the Borel subgroup of G opposite to B with respect to T.

For the notion of a G–linearization, and the GIT quotients, we refer to [MFK, p. 30, p. 40].

Consider the flag variety G/B that parametrizes all Borel subgroups of G. For a character χ of B, let

$$L_\chi = G \times_B \mathbb{C} \longrightarrow G/B$$

be the G–linearized line bundle associated to the action of B on $G \times \mathbb{C}$ given by $b.(g, z) = (gb, \chi(b^{-1})z)$ for $b \in B$ and $(g, z) \in G \times \mathbb{C}$. So, in particular, L_χ is T–linearized. When L_χ is

ample, we denote by $(G/B)^{ss}_T(L_\chi)$ (respectively, $(G/B)^{s}_T(L_\chi)$) the semistable (respectively, stable) locus in G/B for the T–linearized ample line bundle L_χ.

Next we recall some facts about the wonderful compactification of G. Let χ be a regular dominant weight of G with respect to T and B, and let $V(\chi)$ be the irreducible representation of $\widehat{G}$ with highest weight χ, where $\widehat{G}$ is the simply connected covering of G. By [DP, p. 16, 3.4], the wonderful compactification $\overline{G}$, which we denote by X, is the closure of the $G\times G$–orbit of the point

$$[1] \in \mathbb{P}(V(\chi)\otimes V(\chi)^*)$$

corresponding to the identity element 1 of $V(\chi)\otimes V(\chi)^* = \mathrm{End}(V(\chi)^*)$. We denote by $\mathcal{L}_\chi$ the ample line bundle on X induced by this projective embedding. Since the regular dominant weights generate the weight lattice, given a weight χ, we have the line bundle $\mathcal{L}_\chi$ on X associated to χ.

By [DP, Theorem, p. 14, Section 3.1], there is a unique closed $G\times G$–orbit Z in X. Note that

$$Z = \bigcap_{i=1}^{n} D_i\,,$$

where D_i is the $G\times G$ stable irreducible component of $\overline{G}\setminus G$ such that $\mathcal{O}(D_i) = \mathcal{L}_{\alpha_i}$ [DP, p. 29, Section 8.2, Corollary]. Further, Z is isomorphic to $G/B\times G/B^-$ as a $G\times G$ variety. By [DP, p. 26, 8.1], the pullback homomorphism

$$i^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Z)\,,$$

for the inclusion map $i : Z \hookrightarrow X$ is injective and is given by

$$i^*(\mathcal{L}_\chi) = p_1^*(L_\chi)\otimes p_2^*(L_{-\chi}),$$

where L_χ (respectively, $L_{-\chi}$) is the line bundle on G/B (respectively, G/B^-) associated χ (respectively, $-\chi$) and p_j is the projection to the j-th factor of $G/B\times G/B^{-1}$ for $j = 1, 2$.

3. Choice of a polarization on $\overline{G}$

We continue with the notation of Section 2. Let G be a simple algebraic group of adjoint type of rank $n\geq 2$, such that its root system R is different from A_2. Let

$$\mathbb{N}S := \{\sum_{i=1}^{n} m_i\alpha_i : m_i\in\mathbb{N}\}\,.$$

Then, we have the following:

Lemma 3.1. *The above defined $\mathbb{N}S$ contains a regular dominant character χ of T such that $s_{\alpha_i}(\chi)\geq 0$ and $\langle\chi, w(\lambda_i)\rangle\neq 0$ for every $w\in W$ and $1\leq i\leq n$.*

Proof. Denote by $X(T)_{\mathbb{Q}}$ the rational vector space generated by $X(T)$, and also denote by $X(T)^+$ the semi-group of it given by the dominant characters of T. Let $\rho\in X(T)_{\mathbb{Q}}$ be the half sum of positive roots of R. Then, $2\rho = 2(\sum_{i=1}^{n}\omega_i)\in X(T)^+$ is a regular dominant character of T, and we have $2\rho\in\mathbb{N}S$.

Since R is irreducible of rank at-least 2 and different from A_2, we see that for every simple root α_i, there are at-least 3 positive roots β satisfying $\alpha_i \leq \beta$. Hence, the coefficient of every simple root α_j in the expression of $s_{\alpha_i}(2\rho) = 2\rho - 2\alpha_i$ (as a non-negative integral linear combination of simple roots) is positive. Hence, we have $s_{\alpha_i}(2\rho) \in \mathbb{N}S$. Thus, we have

$$2\rho \in X(T)^+ \cap (\bigcap_{i=1}^{n} s_{\alpha_i}(\mathbb{N}S)) .$$

Denote by N the determinant of the Cartan matrix of R. Then we have $N\omega_i \in \mathbb{N}S$ for every $i = 1, 2, \cdots, n$. By the previous discussion, there exists $m \in \mathbb{N}$ such that $ms_{\alpha_i}(2\rho) - N\alpha_i \in \mathbb{N}S$ for every $1 \leq i \leq n$. Hence, we get

$$s_{\alpha_i}(2m\rho + N\omega_i) = ms_{\alpha_i}(2\rho) - N\alpha_i + N\omega_i \in \mathbb{N}S, \ 1 \leq i \leq n ,$$

and from this it follows that

$$2m\rho + N\omega_i \in X(T)^+ \cap (\bigcap_{j=1}^{n} s_{\alpha_j}(\mathbb{N}S)), \ 1 \leq i \leq n.$$

Consider the characters $2m\rho$, $2m\rho + N\omega_2, \cdots$, $2m\rho + N\omega_n$ of T. These are linearly independent in $X(T)$ and by construction they all lie in the rational cone

$$\mathcal{C} \subset X(T)^+_{\mathbb{Q}}$$

generated by the semi-group $X(T)^+ \bigcap (\bigcap_{i=1}^{n} s_{\alpha_i}(\mathbb{N}S))$. It follows that $\mathcal{C}$ has a maximal dimension in $X(T)_{\mathbb{Q}}$, hence it is not contained in any hyperplane of $X(T)_{\mathbb{Q}}$. Therefore, there exists a regular dominant character $\chi \in \mathcal{C} \bigcap \mathbb{N}S$ such that $\langle \chi, w(\lambda_i) \rangle \neq 0$ for all $1 \leq i \leq n$ and every $w \in W$, and hence the lemma follows. $\square$

Lemma 3.2. *Let $\chi \in \mathbb{N}S$ be a regular dominant character of T satisfying the properties stated in Lemma 3.1. Then we have*

(1) $(G/B)^{ss}_T(L_\chi) = (G/B)^s_T(L_\chi)$, *and*
(2) *the set of all unstable points*

$$(G/B) \setminus (G/B)^{ss}_T(L_\chi)$$

is contained in the union of W–translates of all Schubert varieties of codimension at least two.

Proof. Set $L := L_\chi$. Since $\langle \chi, w(\lambda_i) \rangle \neq 0$ for every $w \in W$ and $1 \leq i \leq n$, by [Ka1, p. 38, Lemma 4.1] we have

$$(G/B)^{ss}_T(L) = (G/B)^s_T(L) .$$

This proves (1).

To prove (2), take an unstable point $x \in G/B$ for the polarization L. Then, there is a one-parameter subgroup λ of T such that $\mu^L(x, \lambda) < 0$. Let $\phi \in W$ be such that $\phi(\lambda)$ is in the fundamental chamber, say

$$\phi(\lambda) = \sum_{i=1}^{n} c_i \lambda_i ,$$

where $\{c_i\}$ are non-negative integers. Consequently, we have

$$\mu^L(n_\phi(x),\phi(\lambda)) \,=\, \mu^L(x,\lambda) \,<\, 0\,,$$

where n_ϕ is a representative of ϕ in $N_G(T)$. Now, let $n_\phi(x)$ be in the Schubert cell BwB/B for some $w\in W$. By [Se, Lemma 5.1], we have

$$\mu^L(n_\phi(x),\phi(\lambda)) \,=\, (-\sum_{i=1}^n c_i\langle w(\chi)\,,\lambda_i\rangle) \,<\, 0\,.$$

(The sign here is negative because we are using left action of B on G/B while in [Se, Lemma 5.1] the action of B on $B\backslash G$ is on the right.) Therefore we have $w(\chi) \not\leq 0$. For every $1\leq i\leq n$ we have $s_{\alpha_i}(\chi)\,\geq\, 0$, and hence $w_0 s_{\alpha_i}(\chi)\,\leq\, 0$. Hence we have $l(w_0)-l(w)\,\geq\, 2$. This completes the proof of (2). □

Proposition 3.3. *Let $X \,=\, \overline{G}$ be the wonderful compactification of G. Let χ be as in Lemma 3.2, and let $X^{ss}_T(\mathcal{L}_\chi)$ (respectively, $X^s_T(\mathcal{L}_\chi)$) be the semi-stable (respectively, stable) locus of X for the action of $1\times T$ and the polarization $\mathcal{L}_\chi$ on X. Then we have*

1. *$X^{ss}_T(\mathcal{L}_\chi) \,=\, X^s_T(\mathcal{L}_\chi)$, and*
2. *the set of unstable points $X\setminus(X^{ss}_T(\mathcal{L}_\chi))$ is a union of irreducible closed subvarieties of codimension at least three.*

Proof. Let Z be the unique closed $G\times G$–orbit in X. Let $Z^{ss}_T(\mathcal{L}_\chi)$ (respectively, $Z^s_T(\mathcal{L}_\chi)$) be the semi-stable (respectively, stable) locus of Z for the action of $1\times T$ and the polarization $i^*(\mathcal{L}_\chi)$, where $i\,:\,Z\,\hookrightarrow\, X$ is the inclusion map. Since Z is isomorphic to $G/B\times G/B^-$ and $i^*(\mathcal{L}_\chi)=p_1^*(L_\chi)\otimes p_2^*(L_{-\chi})$, we see that

$$Z^{ss}_T(\mathcal{L}_\chi)\simeq (G/B)\times((G/B^-)^{ss}_T(L_{-\chi}))$$

and $Z^s_T(\mathcal{L}_\chi)\,\simeq\,(G/B)\times((G/B^-)^s_T(L_{-\chi}))$. Set $Z^{ss}\,=\,Z^{ss}_T(\mathcal{L}_\chi)$ and $Z^s\,=\,Z^s_T(\mathcal{L}_\chi)$. By Lemma 3.2 and above discussion, we have $Z^{ss}\,=\,Z^s$.

For convenience, we will denote $X^{ss}_T(\mathcal{L}_\chi)$ and $X^s_T(\mathcal{L}_\chi)$ by X^{ss} and X^s respectively. If $X^{ss}\,\neq\,X^s$, then the complement $X^{ss}\setminus X^s$ is a non-empty $G\times T$ invariant closed subset of X^{ss}. Hence, the complement $(X^{ss}/\!\!/T)\setminus(X^s/\!\!/T)$ is a non-empty $G\times\{1\}$–invariant closed subset of $X^{ss}/\!\!/T$. In particular, $(X^{ss}/\!\!/T)\setminus(X^s/\!\!/T)$ is a finite union of non-empty $G\times\{1\}$–invariant projective varieties. Therefore, there is a $B\times\{1\}$–fixed point in $(X^{ss}/\!\!/T)\setminus(X^s/\!\!/T)$. Let

$$p\in\,(X^{ss}/\!\!/T)\setminus(X^s/\!\!/T)$$

be a $B\times\{1\}$–fixed point. Let Y be the closed $\{1\}\times T$–orbit in the fiber $\pi^{-1}(\{p\})$ over p for the geometric invariant theoretic quotient map $\pi: X^{ss}\,\longrightarrow\, X^{ss}/\!\!/T$. Since this map π is $G\times\{1\}$ equivariant, we conclude that $\pi^{-1}(\{p\})$ is $B\times\{1\}$–invariant. Hence, for any $b\in B$, the translation $(b,1)\cdot Y$ lies in $\pi^{-1}(\{p\})$. Since the actions of $B\times\{1\}$ and $\{1\}\times T$ on X commute with each other, we see that $(b,1)\cdot Y$ is also a closed $\{1\}\times T$–orbit in $\pi^{-1}(\{p\})$. By the uniqueness of the closed $\{1\}\times T$–orbit in $\pi^{-1}(\{p\})$ we conclude that $(b,1)\cdot Y\,=\,Y$. Hence Y is preserved by the action of $B\times\{1\}$. In particular, Y is $U\times\{1\}$–invariant, where $U\,\subset\, B$ is the unipotent radical. The action of $U\times\{1\}$ on

Y induces a homomorphism from U to T/S of algebraic groups, where $\{1\} \times S$ is the stabilizer in $\{1\} \times T$ of some point q in Y. Since there is no nontrivial homomorphism from an unipotent group to a torus, we conclude that $U \times \{1\}$ fixes the point q.

By [DP, p. 32, Proposition], for any regular dominant character χ of T with respect to B, the morphism $X \hookrightarrow \mathbb{P}(V(\chi) \otimes V(\chi)^*)$ is a $G \times G$ equivariant embedding, where $V(\chi)$ is the irreducible representation of G with highest weight χ, and $V(\chi)^*$ is its dual. Hence, the $U \times 1$–fixed point set of X is equal to $X \bigcap \mathbb{P}(\mathbb{C}_\chi \otimes V(\chi)^*)$, where $\mathbb{C}_\chi$ is the one dimensional B–module associated to the character χ. Therefore, by the above discussion, we have $q \in X \bigcap \mathbb{P}(\mathbb{C}_\chi \otimes V(\chi)^*)$.

Further, by [DP, Theorem, p. 30] we have

$$H^0(X, \mathcal{L}_\chi) = \bigoplus_{\nu \leq \chi} V(\nu)^* \otimes V(\nu),$$

where the sum runs over all dominant characters ν of T satisfying $\nu \leq \chi$. By [DP, p. 29, Corollary] and [DP, p. 30, Theorem], the zero locus of

$$\bigoplus_{\nu < \chi} V(\nu)^* \otimes V(\nu) \ \subset\ H^0(X, \mathcal{L}_\chi)$$

in X is the unique closed $G \times G$–orbit $Z = G/B \times G/B^-$. Hence, by the discussion in the previous paragraph, we have $q \in Z$. This contradicts the choice of the polarization $\mathcal{L}_\chi$. Therefore, the proof of (1) is complete.

To prove (2), note that $X \setminus X^{ss}$ is a closed subset of X, and

$$Z \setminus Z^{ss} \ =\ (X \setminus X^{ss}) \cap Z\,.$$

Also, by Lemma 3.2, the complement $Z \setminus Z^{ss} \subset Z$ is of codimension at least two. Since we have $Z = \bigcap_{i=1}^n D_i$, the complement $D_i \setminus D_i^{ss}$ is of codimension at least two for all $1 \leq i \leq n$. Further, every point in the open subset $G \subset X$ is semistable. Hence, $X \setminus X^{ss}$ is of codimension at least three. □

The following lemma will be used in the proof of Corollary 3.5.

Lemma 3.4. *Let H be a reductive algebraic group acting linearly on a polarized projective variety V. Assume that $V^{ss} = V^s$, where V^{ss} (respectively, V^s) is the set of semi-stable (respectively, stable) points of V for the action of H. Then the set of all points in V^{ss} whose stabilizer in H is trivial is actually a Zariski open subset (it may be possibly empty).*

Proof. Consider the morphism

$$f \,:\, H \times V^{ss} \longrightarrow V^{ss} \times V^{ss}\,, \ \ (h\,, v) \longmapsto (h \cdot v\,, v)\,.$$

Since $V^{ss} = V^s$, this map f is proper [MFK, p. 55, Corollary 2.5]. Hence the image

$$M \,:=\, f(H \times V^{ss}) \,\subset\, V^{ss} \times V^{ss}$$

is a closed subvariety. Now, let

$$U' \,\subset\, V^{ss}$$

be the locus of points with trivial stabilizer (for the action of H). Take any

$$v_0 \in U',$$

and set $z_0 := f((1, v_0)) = (v_0, v_0)$. Then, $(f_*\mathcal{O}_{H\times V^{ss}})_{z_0}$ is a free $\mathcal{O}_{M,z_0}$–module of rank one. Hence by [Mu, p. 152, Souped-up version II of Nakayama's Lemma], the locus of points $x \in M$ such that $(f_*\mathcal{O}_{H\times V^{ss}})_x$ is a free $\mathcal{O}_{M,x}$–module of rank at most one is a non-empty Zariski open subset. Since $(f_*(\mathcal{O}_{H\times V^{ss}}))_z$ is nonzero for all $z \in M$, the set of all points $x \in M$ such that $(f_*(\mathcal{O}_{H\times V^{ss}}))_x$ is a free $\mathcal{O}_{M,x}$–module of rank one is a Zariski open subset of M; this Zariski open subset of M will be denoted by U. Note that

$$f^{-1}(U) = p_2^{-1}(U'),$$

where $p_2 : H \times V^{ss} \longrightarrow V^{ss}$ is the second projection. Since p_2 is flat of finite type over $\mathbb{C}$, it is an open map (see [Ha, p. 266, Exercise 9.1]). Hence $U' = p_2(f^{-1}(U))$ is a Zariski open subset. This finishes the proof of the lemma. □

Corollary 3.5. *Let* $X = \overline{\mathrm{PSL}(n+1,\mathbb{C})}$ *be the wonderful compactification of* PSL$(n+1, \mathbb{C})$, $n \geq 3$. *For the choice of the regular dominant character* χ *of* T *as in Proposition 3.3,*

1. *the action of* $\{1\} \times T$ *on* $X^{ss}_T(\mathcal{L}_\chi)/\!\!/T$ *is free,*
2. $X^{ss}_T(\mathcal{L}_\chi)/\!\!/T$ *is a smooth projective embedding of* G/T, *and*
3. *the set of unstable points* $X \setminus (X^{ss}_T(\mathcal{L}_\chi))$ *is a union of irreducible closed subvarieties of codimension at least three.*

Proof. Let χ be a regular dominant character of T as in Proposition 3.3. As in the proof of Proposition 3.3, let Z denote the unique closed $G \times G$ orbit in X. Also, let X^{ss}, X^s, Z^{ss} and Z^s be as in the proof of Proposition 3.3. By Proposition 3.3 we have $X^{ss} = X^s$. Hence by Lemma 3.4, the locus V of points in X^{ss} with trivial stabilizer (for the action of $\{1\} \times T$) is a Zariski open subset of X^{ss}. Therefore, $X^{ss} \setminus V$ is a $G \times \{1\}$ stable closed subvariety of X^{ss}. By using the arguments in the the proof of Proposition 3.3 we see that the set of $B \times \{1\}$–fixed points in $Z \cap (X^{ss} \setminus V)$ is non-empty. But on the other hand by the proof of [Ka2, p. 194, Example 3.3] we see that given any point $z \in Z^{ss}$, its stabilizer subgroup in $\{1\} \times T$ is trivial. This is a contradiction. Hence we conclude that the action of $\{1\} \times T$ on X^{ss} is free. This proves Part (1) and Part (2).

Part (3) follows immediately from the corresponding statement in Proposition 3.3. □

4. Automorphism group of $\overline{\mathrm{PSL}(n+1,\mathbb{C})}^{ss}_T(\mathcal{L})/\!\!/T$

Let $G = \mathrm{PSL}(n+1, \mathbb{C})$, with $n \geq 3$, and define

$$Y := \overline{\mathrm{PSL}(n+1,\mathbb{C})}^{ss}_T(\mathcal{L}_\chi)/\!\!/T,$$

where χ is as in Proposition 3.3.

Theorem 4.1. *Let* A *denote the connected component, containing the identity element, of the group of holomorphic (= algebraic) automorphisms of* Y. *Then*

(1) *A is isomorphic to G, and*
(2) *the Picard group of Y is a free Abelian group of rank $2n$.*

Proof. Let TY denote the algebraic tangent bundle of Y. From [MO, Theorem 3.7] we know that A is an algebraic group. The Lie algebra of A is $H^0(Y,\, TY)$ equipped with the Lie bracket operation of vectors fields.

The Lie algebra of G will be denoted by $\mathfrak{g}$. Define $X := \overline{\mathrm{PSL}(n+1,\mathbb{C})}$ and $U := \overline{\mathrm{PSL}(n+1,\mathbb{C})}^{ss}_T(\mathcal{L}_\chi)$. The connected component, containing the identity element, of the automorphism group of X is $G\times G$ [Br, Example 2.4.5]. From this, and the fact that the complement $X\setminus U \subset X$ is of codimension at least three (see Corollary 3.5), we conclude that

$$H^0(U,\, TU) \,=\, H^0(X,\, TX) \,=\, \mathfrak{g}\oplus\mathfrak{g}\,.$$

Let $\phi : U \longrightarrow Y$ be the geometric invariant theoretic quotient map. Let

$$T_U \,\supset\, T_\phi \,\longrightarrow\, U$$

be the relative tangent bundle for ϕ. Since ϕ makes U a principal T–bundle over Y (see Corollary 3.5(1)), we have the following short exact sequence of vector bundles on U

$$0 \longrightarrow T_\phi \longrightarrow TU \longrightarrow \phi^*(TY) \longrightarrow 0\,, \tag{4.1}$$

and the relative tangent bundle T_ϕ is identified with the trivial vector bundle $\mathcal{O}_U\otimes_{\mathbb{C}}\mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of T.

Set $Z \,=\, X\setminus U$. Since $\mathrm{codim}(Z)\,\geq\, 3$ (Corollary 3.5), we have

$$H^0(U,\, T_\phi) \,=\, H^0(X,\, \mathcal{O}_X\otimes\mathfrak{h}) \,=\, \mathfrak{h}\,.$$

Note that $H^1(U,\, T_\phi) \,=\, H^2_Z(X,\, \mathcal{O}_X\otimes\mathfrak{h})$. Indeed, this follows from the following cohomology exact sequence (see [Gr, Corollary 1.9])

$$H^1(X,\, \mathcal{O}_X\otimes\mathfrak{h}) \longrightarrow H^1(U,\, \mathcal{O}_X\otimes\mathfrak{h}) \longrightarrow H^2_Z(X,\mathcal{O}_X\otimes\mathfrak{h}) \longrightarrow H^2(X,\, \mathcal{O}_X\otimes\mathfrak{h})$$

combined with the fact that $H^i(X,\, \mathcal{O}_X) \,=\, 0$ for all $i\,>\,0$ [DP, p. 30, Theorem]. As X is smooth and $\mathrm{codim}(Z)\,\geq\,3$, it follows from [Gr, Theorem 3.8 and Proposition 1.4] that

$$H^2_Z(X,\, \mathcal{O}_X) \,=\, 0\,,$$

and hence $H^1(U,\, T_\phi) \,=\, 0$. Now, using this fact in the long exact sequence of cohomologies corresponding to the short exact sequence in (4.1) we obtain the following short exact sequence:

$$0 \longrightarrow 0\oplus\mathfrak{h} \longrightarrow \mathfrak{g}\oplus\mathfrak{g} \longrightarrow H^0(U,\, \phi^*TY) \longrightarrow 0\,.$$

Hence, we have

$$H^0(U,\, \phi^*TY) \,=\, \mathfrak{g}\oplus(\mathfrak{g}/\mathfrak{h})\,.$$

By using geometric invariant theory, $H^0(Y,\, TY)$ is the invariant part

$$H^0(Y,\, TY) \,=\, H^0(U,\, \phi^*TY)^{\{1\}\times T} \,\subset\, H^0(U,\, \phi^*TY)\,.$$

Thus we have $H^0(Y,\, TY) \,=\, \mathfrak{g}$. This proves (1).

To prove (2), let $\{D_i \mid 1 \leq i \leq n\}$ be the $(G \times G)$–stable irreducible closed subvarieties of $\overline{G}$ of codimension one such that

$$G = \overline{G} \setminus (\bigcup_{i=1}^{n} D_i)\,.$$

Let $D_i^{ss} = D_i \bigcap X^{ss} \subset D_i$ be the semistable locus of D_i. Set $Z := Y \setminus (G/T)$, and write it as a union

$$Z = \bigcup_{i=1}^{n} Z_i\,,$$

where each $Z_i = D_i^{ss} /\!\!/ T$ is an irreducible closed subvariety of Y of codimension one. As Y is smooth, each Z_i produces a line bundle $L_i \longrightarrow Y$ whose pullback to X^{ss} is $\mathcal{O}_{X^{ss}}(D_i^{ss})$. Since $\text{Pic}(X^{ss}) = \text{Pic}(X)$ and $\{\mathcal{O}_X(D_i)\}_{1 \leq i \leq n}$ are linearly independent in $\text{Pic}(X)$ (see [DP, p. 26, 8.1]), we get that L_i, $1 \leq i \leq n$, are linearly independent in $\text{Pic}(Y)$. The Picard group of G/T is isomorphic to the group of characters of the inverse image $\widehat{T}$ of T inside the simply connected covering $\widehat{G}$ of G (see [KKV]). Now it follows from the exact sequence in [Fu, Proposition 1.8] that $\text{Pic}(Y)$ is a free Abelian group of rank $2n$, thus completing the proof of (2). $\square$

Remark 4.2. The compactification Y of G/T constructed here is an example of a non-spherical variety for the action of G whose connected component of the automorphism group is G.

Remark 4.3. Note that both Y and $G/B \times G/B$ are smooth compactifications of G/T with isomorphic Picard groups. Further both are Fano varieties, i.e., the anti-canonical line bundle is ample. The fact that $G/B \times G/B$ is Fano is well known. That the variety Y is Fano follows as a consequence of the exact sequence in (4.1) together with the facts that X is Fano (see [DP]) and the codimension of $X \setminus U$ is greater than or equal to 3, where X and U are as in the proof of Theorem 4.1. But Y and $G/B \times G/B$ are not isomorphic, as $\text{Aut}^0(Y) \simeq G$ and $\text{Aut}^0(G/B \times G/B) \simeq G \times G$, where $\text{Aut}^0(M)$ denotes the connected component of the group of algebraic automorphisms of a smooth projective variety M containing the identity element.

Remark 4.4. In [St], Strickland extended the construction of $\overline{G}$ to any arbitrary algebraically closed field. Also, $\overline{G}$ is a Frobenius split variety in positive characteristic [St, p. 169, Theorem 3.1] (see [MR] for the definition of Frobenius splitting). Since T is linearly reductive, using Reynolds operator, one can see that the geometric invariant theoretic quotient of $\overline{G}$ for the action of T is also Frobenius split for any polarization on $\overline{G}$.

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